QUIESCENT COSMOLOGICAL SINGULARITIES

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1. INTRODUCTION

This is a report on joint work with Alan Rendall [AR00], on non–oscillatory singularities in four dimensional space–times with scalar field or stiff fluid matter. Before considering the results of [AR00] in detail, I will discuss the BKL proposal which motivated the work.

The singularity theorems of Penrose and Hawking guarantee the existence of spacetime singularities under reasonable assumptions, but give little information about the singularities they predict. The most detailed proposal for the structure of space–time singularities is due to Belinskii, Khalatnikov and Lifshitz (BKL), see [LK63, BKL70, BKL82] and references therein. According to the BKL proposal, singularities in generic four dimensional space–times are space–like and oscillatory, while generic space–times with stiff fluid, including massless scalar fields, have singularities which, according to a proposal of Belinskii and Khalatnikov, see [BK72], are space–like and non–oscillatory. Further, generic space–times will, according to this picture, have the property that spatial points decouple near the singularity, and the local behavior is asymptotically like spatially homogeneous (Bianchi) models locally near the singularity. In particular, in the non–oscillatory case, spatial derivatives become unimportant, the term "asymptotically velocity term dominated" is often used to describe this phenomenon. Space–times with non–stiff matter appear to have the property that asymptotically close to the singularity, matter becomes insignificant and the main features of the evolution is determined by curvature. On the other hand, in the case of stiff matter, the picture is that the matter becomes dominant near the singularity, leading to non–oscillatory behavior. We will refer to the collection of ideas described above as the BKL proposal.

The Bianchi models are space–times with spatially locally homogeneous geometry. This is an interesting toy model for the structure of singularities, and the so–called mixmaster universe, Bianchi IX, has been used as a model for the typical local behavior, near the singularity, of gravity in four dimensions, with non-stiff matter. In the case of Bianchi space–times, the Einstein equations are a system of ODE's and the asymptotic behavior of the space–time at the singularity can be studied using dynamical systems methods. It has recently been proved by H. Ringström [Rin00b] that the singularity in generic Bianchi class A space–times with non–stiff matter, which includes Bianchi IX, is oscillatory. In the stiff fluid or scalar field case, Bianchi IX is non–oscillatory [Rin00a, §20]. In either case, curvature blows up at the singularity, which has as a consequence that the space–time

is inextendible, i.e. the conclusion of the cosmic censorship conjecture cf. [Pen82], holds for Bianchi class A.

For generic space–times, in the case of non–oscillatory singularities, the BKL proposal leads to the space–time being asymptotically Kasner at the singularity, with the Kasner exponents depending on the "point on the singularity". In the oscillatory case, the proposal indicates that locally near the singularity, the space–time returns infinitely often to a state which is approximately a non–flat Kasner. From this it would follow that certain curvature scalars such as the Kretschmann scalar $\kappa = {}^{(4)}R_{\alpha\beta\gamma\delta}{}^{(4)}R^{\alpha\beta\gamma\delta}$ become unbounded near the singularity. Thus, verifying the correctness of the BKL proposal would go a long way towards solving the cosmic censorship problem.

Barrow [Bar78] discussed the structure of the early universe for space–times with stiff matter and used the term "quiescent cosmology" for this picture. Here we will use the term "quiescent singularity" for a singularity with non–oscillatory behavior induced by the presence of matter. In section 2 below I will report on joint work with Alan Rendall [AR00], where we construct families of four dimensional space–times with stiff fluid or scalar field matter, with quiescent singularities. These families have the same number of free functions as the full space of solutions to the Einstein equations, and therefore this result supports the BKL proposal described above for this case.

There are important situations where classes of space–times with symmetry exhibit non–oscillatory behavior at the singularity, even without the presence of stiff matter. An example is the Gowdy class of space–times where both numerical and analytical work gives support to the notion that generic Gowdy space–times have non–oscillatory singularities. Gowdy space–times have a 2–dimensional symmetry group G_2, with the action of G_2 generated by spatial Killing fields with vanishing twist. Rendall and Kichenassamy used Fuchsian methods to construct families, with the "right" number of degrees of freedom, of Gowdy space–times with non–oscillatory singularity. This result uses a singular version of the Cauchy-Kowalewski theorem, and thus requires real analyticity for the data. The Fuchsian method is also used in the four dimensional results described in section 2. The analyticity condition has recently been removed for the Gowdy case by Rendall [Ren00b]. In general, G_2 space–times with matter or with non–vanishing twist have oscillatory singularities [WIB98].

The dimension of the space–time is significant. It has been argued, using a BKL type analysis, that in space–time dimension $D \geq 11$, vacuum gravity generically has non–oscillatory singularity [DHS85]. A low energy limit of string theory leads to consideration of Einstein equations in D dimensions, $D = 10, 11$, with matter consisting of axions, dilatons (scalar field) and form fields. This model is sometimes called "stringy gravity". The possibility that these space–times have non–oscillatory singularities was exploited in the so–called pre–big–bang scenario, where one views the present universe as arising out of a (future) cosmological singularity in a previous universe, the present universe being blown up to a scale determined by the dilaton. This was studied by Buonnanno et al [BDV99] where it was shown using formal expansions that the singularities of stringy gravity in spherical symmetry are non–oscillatory. Families of stringy gravity solutions with Gowdy symmetry and non–oscillatory singularity have been constructed using the Fuchsian method [NTM00]. Recent work by Damour and Henneaux [DH00b, DH00a] indicates that stringy gravity without extra symmetry does in fact have oscillatory singularities.

2. Quiescent singularities

Here we describe the main results of [AR00]. For simplicity we restrict the discussion here to the case of massless scalar field matter. We consider four–dimensional space–times with metric of the form

$$ds^2 = -dt^2 + g_{ab}(t)\theta^a \otimes \theta^b,$$

so that t is a Gaussian time coordinate and $\{\theta^a\}$ is a coframe dual to a frame $\{e_a\}$. In this case, the second fundamental form of a hyper-surface $t =$ constant is given by $k_{ab} = -\frac{1}{2}\partial_t g_{ab}$. The Einstein field equations are

$$^{(4)}R_{\alpha\beta} = 8\pi\nabla_\alpha\phi\nabla_\beta\phi,$$

which can be written in the following equivalent $3+1$ form. The constraints are:

$$R - k_{ab}k^{ab} + (\mathrm{tr}k)^2 = 8\pi[(\partial_t\phi)^2 + \nabla^a\phi\nabla_a\phi],$$
$$\nabla^a k_{ab} - \nabla_b\mathrm{tr}k = -8\pi\partial_t\phi\nabla_b\phi,$$

and the evolution equations are:

$$\partial_t g_{ab} = -2k_{ab},$$
$$\partial_t k^a{}_b = R^a{}_b + (\mathrm{tr}k)k^a{}_b - 8\pi\nabla^a\phi\nabla_b\phi.$$

Here R is the scalar curvature of g_{ab} and R_{ab} its Ricci tensor. It follows from the Einstein–scalar field equations as a consequence of the Bianchi identity that ϕ satisfies the wave equation $^{(4)}g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi = 0$. This has the $3+1$ form:

$$-\partial_t^2\phi + (\mathrm{tr}k)\partial_t\phi + \Delta\phi = 0.$$

The constraints and evolution equations are together equivalent to the full Einstein–scalar field equations.

As mentioned in the introduction, generic non–oscillatory singularities are expected to have the feature that spatial derivatives become unimportant near the singularity. We will make an ansatz for g_{ab}, k_{ab}, ϕ which has leading order terms $^0g_{ab}, {}^0k_{ab}, {}^0\phi$ solving a version of the Einstein evolution equations and the wave equation with (most) spatial derivatives cancelled. This is the velocity dominated system, with constraint equations

$$-{}^0k_{ab}{}^0k^{ab} + (\mathrm{tr}^0k)^2 = 8\pi(\partial_t{}^0\phi)^2,$$
$$\nabla^a({}^0k_{ab}) - e_b(\mathrm{tr}^0k) = -8\pi\partial_t{}^0\phi e_b({}^0\phi),$$

and evolution equations

$$\partial_t{}^0g_{ab} = -2{}^0k_{ab},$$
$$\partial_t{}^0k^a{}_b = (\mathrm{tr}^0k){}^0k^a{}_b - 8\pi\partial_t{}^0\phi^2\delta^a{}_b.$$

The velocity dominated scalar field $^0\phi$ satisfies the equation

$$-\partial_t^2({}^0\phi) + (\mathrm{tr}^0k)\partial_t{}^0\phi = 0.$$

The velocity dominated evolution equations are ordinary differential equations, while the velocity dominated constraint equations still include partial differential equations. The space of solutions of the velocity dominated constraint equations can be proved to have the same number of free functions as the standard Einstein constraint equations.

The solution of the velocity dominated evolution equations is, after an appropriate choice of time, given by ${}^0k^a{}_b = -t^{-1}K^a{}_b$, where $K^a{}_b(x)$ is a time–independend symmetric matrix, which is assumed to have positive eigenvalues $\{p_a(x)\}$, the Kasner exponents. The velocity dominated scalar field ${}^0\phi$ is of the form ${}^0\phi = A\log(t) + B$, where A, B are time–independent functions.

There is an important technical problem when the matrix $K^a{}_b$ has double eigenvalue at some point x. In this case, it is impossible in general to choose a smooth frame $\{e_a\}$ which diagonalizes $K^a{}_b$. This causes some difficulties which which we will gloss over in the discussion below.

Given the solution to the velocity dominated system, we make the following ansatz for the solution of the full 3+1 Einstein scalar field system. We work in terms of a frame $\{e_a\}$ which (approximately) diagonalizes $K^a{}_b$, let q_a be (approximate) eigenvalues of $K^a{}_b$, see [AR00, §5] for details.

For $s \in \mathbb{R}$, let $(s)_+ = \max(s, 0)$, and let α_0 be a small positive number, to be chosen. Let $\alpha^a{}_b = (q_b - q_a)_+ + \alpha_0$. The ansatz we use is

$$g_{ab} = {}^0g_{ab} + {}^0g_{ac}t^{\alpha^c{}_b}\gamma^c{}_b,$$
$$k_{ab} = g_{ac}({}^0k^c{}_b + t^{-1+\alpha^c{}_b}\kappa^c{}_b),$$
$$\phi = {}^0\phi + t^\beta\psi,$$

where $\gamma^c{}_b, \kappa^c{}_b, \psi$ are $o(1)$. The ordo notation used here, and in Theorem 2.1 below, has a precise technical meaning, cf. [AR00, §4]. It is important to note that g_{ab} is not a priori symmetric but this has to be shown as a consequence of the equations.

Rewriting the Einstein and scalar field evolution equations using the ansatz gives, among other equations, the following evolution equations for $\gamma^a{}_b, \kappa^a{}_b$,

$$t\partial_t\gamma^a{}_b + \alpha^a{}_b\gamma^a{}_b + 2\kappa^a{}_b + 2\gamma^a{}_e({}^0k^e{}_b) - 2({}^0k^a{}_e)\gamma^e{}_b = -2t^{\alpha^a{}_e + \alpha^e{}_b - \alpha^a{}_b}\gamma^a{}_e\kappa^e{}_b,$$
$$t\partial_t\kappa^a{}_b + \alpha^a{}_b\kappa^a{}_b - ({}^0k^a{}_b)(\mathrm{tr}\kappa) = t^{\alpha_0}(\mathrm{tr}\kappa)\kappa^a{}_b + t^{2-\alpha^a{}_b}({}^SR^a{}_b - 8\pi g^{ac}e_c(\phi)e_b(\phi)).$$

Here ${}^SR^a{}_b$ is the Ricci tensor with respect to a symmetrized version of g_{ab}. It can be shown that the power of t occurring on the right hand side of the evolution equation for $\gamma^a{}_b$ is positive.

In order to write the system in first order form, it is necessary to introduce an additional field $\lambda^a{}_{bc}$, defined in terms of the first order spatial derivatives of $\gamma^a{}_b$. Proceeding in the same way for the wave equation results in a first order system in the variables ψ, ω_a, χ, where ω_a, χ encodes first order space and time derivatives of ψ, respectively. Let $u = (\gamma^a{}_b, \lambda^a{}_{bc}, \kappa^a{}_b, \psi, \omega_a, \chi)$ be the unknown in the resulting first order system. The Einstein scalar field system then takes the Fuchsian form

$$(2.1) \qquad\qquad\qquad t\partial_t u + AU = f[t, x, u, u_x],$$

where $f = o(1)$, in the sense that f applied to the fields specified by the ansatz tends to zero as a power of t, as t tends to zero, and the matrix function A satisfies a spectral condition, related to being positive definite, cf. [AR00, §4]. Under an analyticity condition, existence and uniqueness for the system (2.1) holds by a singular version of the Cauchy–Kowalewski theorem, cf. [AR00, Theorem 3]. The main technical difficulty in proving that the Einstein–scalar field system is in the Fuchsian form (2.1) lies in the proof that the function f is $o(1)$ in the appropriate sense. For the proof that f is $o(1)$, the most important point is to estimate the term $t^{2-\alpha^a{}_b}{}^SR^a{}_b$ occurring in the evolution equation for $\kappa^a{}_b$. This estimate requires a detailed analysis of the blowup rate of the connection forms of the metric g_{ab},

and their derivatives. The Einstein–scalar field and Einstein–stiff fluid systems can be written in Fuchsian form only if $p_a > 0$, in the four–dimensional case we are considering. The above proves existence and uniqueness of solutions up to the singularity, for the Einstein–scalar field system with real analytic data.

Theorem 2.1 ([AR00, Theorem 1]). *Let S be a real analytic manifold of dimension 3, and let $({}^0g_{ab}(t), {}^0k_{ab}(t), {}^0\phi(t))$ be a real analytic solution of the velocity dominated Einstein-scalar field equations on $S \times (0, \infty)$, such that $t\mathrm{tr}^0k = -1$ and $-t^0k^a{}_b$ has positive eigenvalues. Then there exists an open neighbourhood U of $S \times \{0\}$ in $S \times [0, \infty)$ and a unique real analytic solution $(g_{ab}(t), k_{ab}(t), \phi(t))$ of the Einstein-scalar field equations on $U \cap (S \times (0, \infty))$ such that for each compact subset $K \subset S$ there are positive real numbers $\zeta, \beta, \alpha^a{}_b$, for which the following estimates hold uniformly on K:*

1. ${}^0g^{ac}g_{cb} = \delta^a{}_b + o(t^{\alpha^a{}_b})$,
2. $k^a{}_b = {}^0k^a{}_b + o(t^{-1+\alpha^a{}_b})$,
3. $\phi = {}^0\phi + o(t^\beta)$,
4. $\partial_t \phi = \partial_t{}^0\phi + o(t^{-1+\beta})$,
5. ${}^0g^{ac}e_f(g_{cb}) = o(t^{\alpha^a{}_b - \zeta})$,
6. $e_a(\phi) = e_a({}^0\phi) + o(t^{\beta - \zeta})$.

The positivity condition on the eigenvalues together with the velocity dominated Hamiltonian constraint imply that the function A occurring in ${}^0\phi$ has the property that A^2 is strictly positive in the velocity dominated solution. Thus vacuum solutions are ruled out by the hypotheses of this theorem. See [AR00, Theorem 2] for an analogous result in the stiff fluid case. See also [Ren00a] for a discussion of the case of Einstein equations coupled to a nonlinear scalar field. It is an interesting problem to generalize the results discussed here to C^∞ data, as was done for the Gowdy problem in [Ren00b].

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